

Absolute Nevanlinna Summability and Fourier Series

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We adopt here an extended version of the absolute Nevanlinna summability and apply it to study Fourier series of functions of bounded variations. The absolute Riesz summability $|R, n, \gamma|$, $\gamma \geq 0$, which is equivalent to the absolute Cesàro summability $|C, \gamma|$, is obtainable from the Nevanlinna summability. As such from the theorems proved here we deduce some results on the absolute Cesàro summability of Fourier series. Some of these results are new while some others improve upon known theorems. © 2000 Academic Press

1. INTRODUCTION

1.1. Nevanlinna [12] initiated an interesting function-to-function transformation N_q and adopted it for a study of Fourier series. Moursund [10, 11] adopted the Nevanlinna transform and extended it further by introducing a parameter to study Fourier series and differentiated Fourier series. Ray and Samal [14] introduced the concept of the absolute Nevanlinna summability $|N_q|$ to study the summability of Fourier series and the conjugate Fourier series. Later on Samal [15] and Ray and Sahoo [13] extended the method in different directions to study the absolute summability of some Fourier associated series.

We adopt here an extension of the method $|N_q|$ which is on the pattern followed by Moursund [11] to extend the method N_q of Nevanlinna. We shall denote the method by $|N(q_\delta)|$. We apply the method $|N(q_\delta)|$ to the study of Fourier series. Moursund's extension covers the Riesz method (R, n, γ) , $\gamma > 0$. The method $|N(q_\delta)|$ similarly yields the Riesz method $|R, n, \gamma|$, $\gamma > 0$, which is known to be equivalent to the Cesàro-method $|C, \gamma|$. Results proved for the method $|N(q_\delta)|$ may thus relate to the results on absolute Cesàro summability.



DEFINITION 1.1. Given a series $\sum u_n$ let $F(w) = \sum_{n < w} u_n$. Let $q_\delta \equiv q_\delta(t)$ be defined for $0 \leq t < 1$. The $N(q_\delta)$ transform $N(F, q_\delta)$ of F is defined by

$$N(F, q_\delta)(w) = \int_0^1 q_\delta(t) F(wt) dt.$$

The series $\sum u_n$ is said to be summable by the method $N(q_\delta)$ to the sum s if

$$\lim_{w \rightarrow \infty} N(F, q_\delta)(w) = s.$$

It is said to be absolutely summable by the method $N(q_\delta)$ and we shall write $\sum u_n \in |N(q_\delta)|$ if

$$N(F, q_\delta)(w) \in BV(A, \infty)$$

for some $A \geq 0$, which is indeed equivalent to

$$\int_A^\infty \left| \sum_{n < w} q_\delta\left(\frac{n}{w}\right) nu_n \right| \frac{dw}{w^2} < \infty \quad (1)$$

(see [14, 15]). For the regularity we need (see [12, 14, 15])

$$\int_0^1 q_\delta(t) dt = 1. \quad (2)$$

The parameter δ will be a non-negative real number. We have further two sets of restrictions on q_δ : one for $0 \leq \delta < 1$ and the other for $\delta \geq 1$.

$$\text{In the case } 0 \leq \delta < 1, q_\delta(t) \text{ is increasing for } 0 < t < 1. \quad (3)$$

In the case $\delta \geq 1$, q_δ satisfies (3a)–(3d):

$$q_\delta(t) \text{ is decreasing for } 0 < t < 1 \quad (3a)$$

with $p = [\delta]$, the integral part of δ ,

$$\left(\frac{d}{dt}\right)^{p-1} q_\delta(t) \in AC[0, 1] \quad (3b)$$

$$\left[\left(\frac{d}{dt}\right)^k q_\delta(t)\right]_{t=1} = 0, \quad k = 0, 1, \dots, p-1 \quad (3c)$$

$$(-1)^p \left(\frac{d}{dt}\right)^p q_\delta(t) \geq 0 \text{ and is increasing.} \quad (3d)$$

Also, for $\delta \geq 0$, $p = [\delta]$, we assume

$$Q_\delta(t)/t^{\delta-p+1} \in L(0, 1), \quad (4)$$

where $Q_\delta(t) := \int_{1-t}^1 q_\delta^{(p)}(x) dx$.

1.2. *Notations.* Let $f \in L(-\pi, \pi)$ and let it be 2π -periodic with its Fourier series given by

$$f(x) \sim \frac{1}{2}a_0 + \sum_1^\infty (a_n \cos nx + b_n \sin nx) \equiv \sum_0^\infty A_n(x).$$

We shall adhere to the notations

$$\begin{aligned} \phi(t) &= \frac{1}{2}\{f(x+t) + f(x-t)\} \\ \Phi_0(t) &= \phi(t), \end{aligned}$$

and for $\alpha > 0$

$$\Phi_\alpha(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-u)^{\alpha-1} \phi(u) du$$

$$\phi_\alpha(t) = \Gamma(\alpha+1)t^{-\alpha}\Phi_\alpha(t), \quad \alpha \geq 0$$

$$S(w, t, p, r) = \sum_{n < w} (w-n)^r n^p \cos(nt + \theta), \quad \theta \text{ independent of } n$$

$$g(n, t) = \int_t^\pi (y-t)^{h-\alpha} \cos(ny - \frac{1}{2}h\pi) dy,$$

where

$$h = [\alpha], \quad \text{the integral part of } \alpha$$

$$H^*(n, t, a) = \frac{1}{\Gamma(\alpha+1)} \int_t^\pi v^{\alpha+a} \frac{d}{dv} g(n, v) dv$$

and

$$H(n, t) = H^*(n, t, 0).$$

We may assume without any loss in generality that $A_0 = 0$.

C, C_0, C_1, \dots will denote constants which will be known in the context and K, K_0, K_1, \dots will stand for absolute constants. These constants will probably be different at successive occurrences.

We may find it helpful to introduce empty sums each of which will represent a zero.

2. THEOREMS

We shall first prove the following theorem on $|N(q_\delta)|$ summability of Fourier series at a point.

THEOREM 2.1. *Let $\alpha \geq 0$ and let the functions q_α satisfy the conditions (2)–(4) with $\delta = \alpha$. If $\phi_\alpha(t) \in BV(0, \pi)$, then at $t = x$ the Fourier series of f is summable by the method $|N(q_\alpha)|$.*

Remarks. (1) The theorem for $\alpha = 0$ is due to Ray and Samal [14] (see also Samal [15]).

(2) The kernel $q_\alpha(t) = (\alpha + \epsilon)(1 - t)^{\alpha + \epsilon - 1}$, $1 > \epsilon > 0$, meets the said requirements of the theorem. In this case the method reduces to the Riesz method $|R, n, \alpha + \epsilon|$, which is equivalent to the Cesàro method $|C, \alpha + \epsilon|$. We thus deduce the following classical theorem of Bosanquet:

THEOREM A [1, 2]. *Let $\alpha \geq 0$ and $\epsilon > 0$. If $\phi_\alpha(t) \in BV(0, \pi)$ then the Fourier series $\Sigma A_n(x)$ is summable $|C, \alpha + \epsilon|$.*

(3) The following lemma shows that our theorem provides an improvement on Bosanquet's result.

LEMMA 2.2. *If kernel functions q_a and q_b are such that $q = q_b/q_a \in BV(0, 1)$ then $|N(q_a)| \subset |N(q_b)|$.*

Proof. We have indeed

$$\begin{aligned}
 \int_A^\infty |dN(f, q_b)| &= \int_A^\infty \int_0^1 q_b(t) \left| \frac{\partial}{\partial w} F(wt) \right| dt dw \\
 &= \int_A^\infty \int_0^1 q(t) q_a(t) \left| \frac{\partial}{\partial w} F(wt) \right| dt dw \\
 &\leq \int_A^\infty \left[q(u) \int_0^u q_a(t) \left| \frac{\partial}{\partial w} F(wt) \right| dt \right]_{u=0}^1 dw \\
 &\quad + \int_A^\infty \int_0^1 |q'(u)| \int_0^u q_a(t) \left| \frac{\partial}{\partial w} F(wt) \right| dt du dw \\
 &= q(1) \int_A^\infty \int_0^1 q_a(t) \left| \frac{\partial}{\partial w} F(wt) \right| dt dw \\
 &\quad + \int_A^\infty \int_0^1 q_a(t) \left| \frac{\partial}{\partial w} F(wt) \right| \left| \int_t^1 |q'(u)| du \right| dt dw \\
 &\leq K \int_A^\infty \int_0^1 q_a(t) \left| \frac{\partial}{\partial w} F(wt) \right| dt dw \\
 &= K \int_A^\infty |dN(F, q_a)|.
 \end{aligned}$$

Let a function q^* be defined over $[0, 1]$ and let $q_\alpha(t) = (1 - t)^{\alpha-1}q^*(t)$. If q^* is such that for $\beta > \alpha$

$$\frac{(1 - t)^{\beta-\alpha}}{q^*(t)} \in BV[0, 1],$$

and $q_\alpha(t)$ satisfies the requirements of the method $|N(q_\alpha)|$ for the theorem then we obtain a refinement of Bosanquet's Theorem.

The study of absolute summability of Fourier series has also been done for several methods of summation under varied hypotheses on ϕ_α . For example, for results involving $t^{-a}\phi_\alpha(t) \in BV(0, \pi)$ one may refer to: (i) Dikshit [6] for the Euler method $|E, q|$ and (ii) Dikshit [5] and Dikshit and Kuttner [7] involving Riesz summability of different "types" determined by a class of logarithmico-exponential functions.

The absolute summability of Fourier series is also studied under hypotheses involving

$$\int_0^\pi t^{-c} |d\phi_\alpha(t)| < \infty.$$

For Nevanlinna summability Samal gave the following theorem for the case $\alpha = 0$:

THEOREM B [15]. *Let $1 > c > 0$. Let the function q_c satisfy the conditions (2), and (3) and let $Q_c(t)/t^{c+1} \in L(0, 1)$. Then*

$$\int_0^\pi t^{-c} |d\phi(t)| < \infty \Rightarrow \Sigma n^c A_n(x) \in |N(q_c)|.$$

This theorem then includes as a special case a theorem of Mohanty on Cesàro summability $|C, k|$.

In this paper we pick up also the two types of above named hypotheses on ϕ_α to study the extended Nevanlinna summability of some factored Fourier series and give two theorems. Corollaries to our Theorem 2.4 dealing with the Cesàro method extend the theorem of Mohanty and provide results which are much sharper than that of Matsumoto [8]. Similarly a corollary to Theorem 2.3 furnishes better summability factors than provided in Dikshit [5] for summability $|C, k|$.

We shall prove the following theorems:

THEOREM 2.3. *Let $1 > \alpha \geq 0$ and a and b be real numbers such that $(\alpha + a) \geq 0$ and (i) $-1 \leq b < a < 1$, or (ii) when $\alpha = 0$, $a = 0$, $a \geq b \geq -1$.*

Let the function q_α for the summability $|N(q_\alpha)|$ satisfy the hypotheses (2) and (3) and let

$$Q_\alpha(t)/t^{\alpha+b+1} \in L(0,1).$$

Then

$$t^{-a}\phi_\alpha(t) \in BV(0, \pi) \Rightarrow \Sigma n^b A_n(x) \in |N(q_\alpha)|.$$

THEOREM 2.4. Let $1 > \alpha \geq 0$ and b and c be real numbers such that $1 > c \geq -1$, $(\alpha + c) \geq 0$, and $c \geq b$ unless $b = -\alpha \neq 0$ in which case $c > b$. Let the function q_α of the summability $|N(q_\alpha)|$ satisfy the hypotheses (2) and (3) and let

$$Q_\alpha(t)/t^{\alpha+b+1} \in L(0,1).$$

Then

$$\int_0^\pi t^{-c} |d\phi_\alpha(t)| < \infty \Rightarrow \Sigma n^b A_n(x) \in |N(q_\alpha)|.$$

Remarks. We may note that after the Riemann–Lebesgue Lemma a proof for Theorem 2.3 and Theorem 2.4 needs to be discussed only for the cases $b \geq -1$.

3. LEMMAS

We shall make use of the results in the following lemmas. Hence onward we shall assume that q satisfies the regularity conditions (2)–(4), with δ replaced by α .

LEMMA 3.1 [11]. Let $\alpha \geq 1$ and $0 \leq k \leq h - 1$. Then $(-1)^k (\frac{d}{dt})^k q_\alpha(t)$ is decreasing over $[0, 1]$.

LEMMA 3.2. (i) [15]. Let $\alpha - h > 0$. Then $Q_\alpha(t)/t^{1+\alpha-h} \in L(0,1)$ if, and only if

$$q_\alpha^{(h)}(t)/(1-t)^{\alpha-h} \in L(0,1).$$

(ii) [11]. Let $\alpha = h \geq 0$. Then $Q_\alpha(t)/t \in L(0,1)$, if and only if

$$q_\alpha^{(h)}(t) \log \left(\frac{1}{1-t} \right) \in L(0,1).$$

In particular, the hypothesis (4) with $\delta = \alpha$ gives that $q_\alpha^{(h)}(t) \in L(0,1)$.

LEMMA 3.3 [4]. *Let p be a positive integer. Then for $r = p$, or $r = p - 1$ and $0 < t \leq \pi$,*

$$S(w, t, p, r) = O(w^p t^{-r-1}).$$

LEMMA 3.4. *Let $\alpha \geq 1$ and $\frac{\pi}{w} < t \leq v \leq \pi$. Then*

- (i) *for $s = 0, 1, \dots, (h-1)$, $\sum_{n < w} q_\alpha\left(\frac{n}{w}\right) n^s \cos(nv + \theta) = O(v^{-s-1})$*
- (ii) $\sum_{n < w} q_\alpha\left(\frac{n}{w}\right) n^h \cos(nv + \theta) = O(v^{-h-1} q_\alpha^{(h)}(1 - \frac{\pi}{wt}))$
 $+ O(wv^{-h} Q_\alpha(\frac{\pi}{tw}))$
- (iii) $\sum_{n < w} q_\alpha\left(\frac{n}{w}\right) n^{h+1} \cos(nv + \theta) = O(wv^{-h-1} q_\alpha^{(h)}(1 - \frac{\pi}{wt}))$
 $+ O(w^2 v^{-h} Q_\alpha(\frac{\pi}{tw}))$.

Proof. (i) As q_α is decreasing the result for $s = 0$ is rather trivial. For $1 \leq s \leq h-1$, using Lemma 3.3 we obtain

$$\begin{aligned} & \sum_{n < w} q_\alpha\left(\frac{n}{w}\right) n^s \cos(nv + \theta) \\ &= q_\alpha(1) \sum_{n < w} n^s \cos(nv + \theta) - \int_0^w S(u, v, s, 0) \frac{\partial}{\partial u} q_\alpha\left(\frac{u}{w}\right) du \\ &= -\frac{1}{s!} \int_0^w \left(\frac{\partial}{\partial u}\right)^s S(u, v, s, s) \frac{\partial}{\partial u} q_\alpha\left(\frac{u}{w}\right) du \\ &= \frac{1}{s!} \sum_{r=1}^s (-1)^r \left(\frac{\partial}{\partial u}\right)^r q_\alpha\left(\frac{u}{w}\right) \left(\frac{\partial}{\partial u}\right)^{s-r} S(u, v, s, s) \Big|_{u=0}^w \\ &\quad + \frac{1}{s!} \int_0^w (-1)^{s+1} \left(\frac{\partial}{\partial u}\right)^{s+1} q_\alpha\left(\frac{u}{w}\right) S(u, v, s, s) du \\ &= O(v^{-s-1}) \int_0^w u^s \left| \left(\frac{\partial}{\partial u}\right)^{s+1} q_\alpha\left(\frac{u}{w}\right) \right| du \\ &= O(v^{-s-1}) \int_0^1 y^s |q_\alpha^{(s+1)}(y)| dy \\ &= O(v^{-s-1}). \end{aligned}$$

(ii) Proceeding as above we have

$$\begin{aligned}
 & \sum_{n < w} q_\alpha \left(\frac{n}{w} \right) n^h \cos(nv + \theta) \\
 &= -\frac{1}{h!} \int_0^w \left(\frac{\partial}{\partial u} \right)^h S(u, v, h, h) \frac{\partial}{\partial u} q_\alpha \left(\frac{u}{w} \right) du \\
 &= \frac{1}{h!} \sum_{r=1}^{h-1} (-1)^r \left(\frac{\partial}{\partial u} \right)^r q_\alpha \left(\frac{u}{w} \right) \left(\frac{\partial}{\partial u} \right)^{h-r} S(u, v, h, h) \Bigg|_{u=0}^w \\
 &\quad + \frac{1}{h!} \int_0^w (-1)^h \left(\frac{\partial}{\partial u} \right)^h q_\alpha \left(\frac{u}{w} \right) \frac{\partial}{\partial u} S(u, v, h, h) du \\
 &= \frac{1}{h!} \left(\int_0^{w-\pi/t} + \int_{w-\pi/t}^w \right) (-1)^h \left(\frac{\partial}{\partial u} \right)^h q_\alpha \left(\frac{u}{w} \right) \left(\frac{\partial}{\partial u} \right) S(u, v, h, h) du \\
 &= O \left(w^{-h} \left| (-1)^h q_\alpha^{(h)} \left(1 - \frac{\pi}{tw} \right) \right| \left\| \sup_{0 \leq u \leq w-\pi/t} S(u, v, h, h) \right\| \right. \\
 &\quad \left. + O \left(\int_{w-\pi/t}^w \left| \left(\frac{\partial}{\partial u} \right)^h q_\alpha \left(\frac{u}{w} \right) \right| u^h v^{-h} du \right) \right) \\
 &= O \left(v^{-h-1} \left| q_\alpha^{(h)} \left(1 - \frac{\pi}{tw} \right) \right| \right) + O \left(v^{-h} \int_{1-\pi/tw}^1 w y^h |q_\alpha^{(h)}(y)| dy \right) \\
 &= O \left(v^{-h-1} \left| q_\alpha^{(h)} \left(1 - \frac{\pi}{tw} \right) \right| \right) + O \left(w v^{-h} Q_\alpha \left(\frac{\pi}{tw} \right) \right).
 \end{aligned}$$

(iii) Next, once again we obtain

$$\begin{aligned}
 & \sum_{n \leq w} q_\alpha \left(\frac{n}{w} \right) n^{h+1} \cos(nv + \theta) \\
 &= -\frac{1}{h!} \int_0^w \left(\frac{\partial}{\partial u} \right)^h S(u, v, h, h) \frac{\partial}{\partial u} \left(u q_\alpha \left(\frac{u}{w} \right) \right) du \\
 &= \frac{1}{h!} \int_0^w (-1)^h \left\{ u \left(\frac{\partial}{\partial u} \right)^h q_\alpha \left(\frac{u}{w} \right) + h \left(\frac{\partial}{\partial u} \right)^{h-1} q_\alpha \left(\frac{u}{w} \right) \right\} \\
 &\quad \times \frac{\partial}{\partial u} S(u, v, h, h) du
 \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{h!} \left\{ \int_0^{w-\pi/t} + \int_{w-\pi/t}^w \right\} u(-1)^h \left(\frac{\partial}{\partial u} \right)^h q_\alpha \left(\frac{u}{w} \right) \frac{\partial}{\partial u} S(u, v, h, h) du \\
&\quad + \frac{1}{h!} \left[h(-1)^h \left(\frac{\partial}{\partial u} \right)^{h-1} q_\alpha \left(\frac{u}{w} \right) S(u, v, h, h) \right]_{u=0}^w \\
&\quad - \frac{h}{h!} \left\{ \int_0^{w-\pi/t} + \int_{w-\pi/t}^w \right\} (-1)^h \left(\frac{\partial}{\partial u} \right)^h q_\alpha \left(\frac{u}{w} \right) S(u, v, h, h) du \\
&= O \left(w v^{-h-1} \left| q_\alpha^{(h)} \left(1 - \frac{\pi}{tw} \right) \right| \right) + O \left(w^2 v^{-h} Q_\alpha \left(\frac{\pi}{tw} \right) \right),
\end{aligned}$$

on proceeding from part (ii) and noticing that $\frac{\pi}{w} < v$.

This completes the proof of the lemma.

LEMMA 3.5. *Let $\alpha > 0$ and $0 < t \leq \pi$. Then*

- (i) $H(n, 0) = \sum_{r=0}^{h-1} C_r n^{-r-1} \cos(n - \frac{1}{2}(h-r-1)\pi)$,
- (ii) $H(n, t) = H(n, 0) + O(t^\alpha n^{\alpha-h-1})$, and also
- (iii) $|H(n, t)| = K_1 |H(n, 0)| + K_2 t^\alpha |g(n, t)| + \sum_{r=0}^h K_{r+3} n^{-r-1} t^{h-r} \left| \int_{t/\pi}^1 x^{\alpha-h+r-1} (1-x)^{h-\alpha} \cos(\frac{nt}{x} - \frac{1}{2}(h-r-1)\pi) dx \right|$.

These results are rather routine and are familiar. For the sake of completeness we write a proof.

Proof.

$$\Gamma(\alpha + 1) H(n, t)$$

$$\begin{aligned}
&= [v^\alpha g(n, v)]_t^\pi - \alpha \int_t^\pi v^{\alpha-1} \int_v^\pi (y-v)^{h-\alpha} \cos(ny - \frac{1}{2}h\pi) dy dv \\
&= -t^\alpha g(n, t) - \alpha \int_t^\pi \cos(ny - \frac{1}{2}h\pi) \int_t^y v^{\alpha-1} (y-v)^{h-\alpha} dv dy \\
&= -t^\alpha g(n, t) - \alpha \int_t^\pi y^h \cos(ny - \frac{1}{2}h\pi) \int_{t/y}^1 x^{\alpha-1} (1-x)^{h-\alpha} dx dy \\
&= -t^\alpha g(n, t) - \alpha \int_{t/\pi}^1 x^{\alpha-1} (1-x)^{h-\alpha} \int_{t/s}^\pi y^h \cos(ny - \frac{1}{2}h\pi) dy dx \\
&= -t^\alpha g(n, t) \\
&\quad - \alpha \int_{t/\pi}^1 \left[\sum_{r=0}^h C_r n^{-r-1} y^{h-r} \cos(ny - \frac{1}{2}(h-r-1)\pi) \right]_{y=t/x}^\pi \\
&\quad x^{\alpha-1} (1-x)^{h-\alpha} dx.
\end{aligned} \tag{5}$$

Hence with $t = 0$ in (5) we get

$$H(n, 0) = \sum_{r=0}^{h-1} C_r n^{-r-1} \cos\left(n - \frac{1}{2}(h-r-1)\pi\right). \quad (6)$$

Next, we have

$$\begin{aligned} & \Gamma(\alpha + 1)H(n, t) \\ &= \Gamma(\alpha + 1)H(n, 0) - \int_0^t v^\alpha \frac{d}{dv} g(n, v) dv \\ &= \Gamma(\alpha + 1)H(n, 0) - t^\alpha g(n, t) + \alpha \int_0^t v^{\alpha-1} g(n, v) dv, \end{aligned} \quad (7)$$

where, in the case $v + \frac{1}{n} < \pi$,

$$\begin{aligned} |g(n, v)| &= \left| \left(\int_v^{v+1/n} + \int_{v+1/n}^\pi \right) (y-v)^{h-\alpha} \cos(ny - \tfrac{1}{2}h\pi) dy \right| \\ &\leq \int_v^{v+1/n} (y-v)^{h-\alpha} dy \\ &\quad + \left| \left[(y-v)^{h-\alpha} n^{-1} \sin(ny - \tfrac{1}{2}h\pi) \right]_{v+1/n}^\pi \right. \\ &\quad \left. - (h-\alpha)n^{-1} \int_{v+1/n}^\pi (y-v)^{h-\alpha-1} \sin(ny - \tfrac{1}{2}h\pi) dy \right| \\ &= O(n^{\alpha-h-1}). \end{aligned} \quad (8)$$

The result at (8) is trivially true if $v \leq \pi \leq v + \frac{1}{n}$.

The required estimates now follow from (5)–(8).

LEMMA 3.6. *Let $1 > \alpha \geq 0$, $\rho \geq -1$ and $(2\pi/w) < t \leq v \leq \pi$. Then*

$$\begin{aligned} & \sum_{n < w-1} q_\alpha(n/w) n^\rho \cos(nv + \theta) \\ &= O(q_\alpha(1 - \pi/tw)) \{ v^{-1} (K_1 w^\rho - K_2 v^{-\rho}) + K_3 \log(\pi/v) \} \\ &\quad + O(w^{\rho+1}) (K_4 + K_5 (1 - \pi/tw)^\rho) Q_\alpha(\pi/tw). \end{aligned}$$

Proof. Let

$$\sum_{n < w-1} q_{\alpha}(n/w) n^{\rho} \cos(nv + \theta) = \sum_{n \leq (w - \pi/t)} + \sum_{(w - \pi/t) < n < w-1}.$$

First we note that

$$\begin{aligned} & \left| \sum_{(w - \pi/t) < n < w-1} q_{\alpha}(n/w) n^{\rho} \cos(nv + \theta) \right| \\ & \leq K \int_{w - \pi/t}^w q_{\alpha}(u/w) u^{\rho} du \\ & = Kw^{\rho+1} \int_{1 - \pi/tw}^1 q_{\alpha}(z) z^{\rho} dz \\ & = (K_1 + K_2(1 - \pi/tw)^{\rho}) w^{\rho+1} Q_{\alpha}(\pi/tw). \end{aligned} \quad (9)$$

Next by Abel's lemma, for $\rho \geq 0$

$$\sum_{n < (w - \pi/t)} q_{\alpha}(n/w) n^{\rho} \cos(nv + \theta) = O(w^{\rho} q_{\alpha}(1 - \pi/tw) v^{-1}). \quad (10)$$

For $-1 \leq \rho < 0$, writing

$$\begin{aligned} & \left| \sum_{k \leq (w - \pi/t)} k^{\rho} \cos(kv + \theta) \right| \\ & = \sum_{k < \pi/v} + \sum_{\pi/v \leq k \leq (w - \pi/t)} \\ & = O\left(\int_1^{\pi/v} y^{\rho} dy\right) + (\pi/v)^{\rho} \sup_{0 < v < \pi} \left| \sum_k \cos(kv + \theta) \right| \\ & = O(v^{-\rho-1}) + O(\log \pi/v), \end{aligned}$$

we obtain

$$\begin{aligned} & \left| \sum_{n < (w - \pi/t)} q_{\alpha}(n/w) n^{\rho} \cos(nv + \theta) \right| \\ & \leq q_{\alpha}(1 - \pi/tw) \sup_{0 < v < \pi} \left| \sum_k k^{\rho} \cos(kv + \theta) \right| \\ & = O(v^{-\rho-1} q_{\alpha}(1 - \pi/tw)) + O(\log(\pi/v) q_{\alpha}(1 - \pi/tw)). \end{aligned} \quad (11)$$

Collecting the estimates from (9)–(11) we complete the proof of the lemma.

LEMMA 3.7. *Let $1 > \alpha \geq 0$, $-1 < a < 1$, $(\alpha + a) \geq 0$, and $0 \leq t \leq \pi$. Then*

- (i) *for $\alpha = 0$, $a = 0$, $H^*(n, t, 0) = (\sin nt)/n$*
- (ii) *for $a \neq 0$, $H^*(n, 0, a) = O(n^{-\alpha-1})$*
- (iii) *$H^*(n, t, a) = H^*(n, 0, a) + O(t^{\alpha+a}n^{\alpha-1})$ and also*
- (iv) *$H^*(n, t, a) = -t^{\alpha+a}g(n, t) - (\alpha + a)\int_{t/\pi}^1 x^{\alpha+a-1}(1-x)^{-\alpha} \times \int_{t/x}^\pi y^a \cos ny dy dx$, where the second term in (iv) is zero in the case $(\alpha + a) = 0$.*

Proof. We have

$$\Gamma(\alpha + 1)H^*(n, t, a) = \int_t^\pi v^{\alpha+a} (d/dv) \int_v^\pi (y-v)^{-\alpha} \cos ny dy dv.$$

Hence for $\alpha = 0$

$$H^*(n, t, a) = -\int_t^\pi v^a \cos nv dv$$

and so for $\alpha = 0$, $a = 0$

$$H^*(n, t, 0) = (\sin nt)/n.$$

Also when $(\alpha + a) = 0$ and $a \neq 0$,

$$\Gamma(\alpha + 1)H^*(n, t, a) = -g(n, t).$$

For $(\alpha + a) > 0$

$$\begin{aligned} \Gamma(\alpha + 1)H^*(n, t, a) &= [v^{\alpha+a}g(n, v)]_t^\pi - (\alpha + a)\int_t^\pi v^{\alpha+a-1} \int_v^\pi (y-v)^{-\alpha} \cos ny dy dv \\ &= -t^{\alpha+a}g(n, t) - (\alpha + a)\int_t^\pi \cos ny \int_t^y v^{\alpha+a-1}(y-v)^{-\alpha} dv dy \\ &= -t^{\alpha+a}g(n, t) - (\alpha + a)\int_t^\pi y^a \cos ny \int_{t/y}^1 x^{\alpha+a-1}(1-x)^{-\alpha} dx dy \\ &= -t^{\alpha+a}g(n, t) - (\alpha + a)\int_{t/\pi}^1 x^{\alpha+a-1}(1-x)^{-\alpha} \int_{t/x}^\pi y^a \cos ny dy dx, \end{aligned}$$

which is the result at (iv).

Further for $(\alpha + a) \geq 0$ when $t = 0$ we get

$$H^*(n, 0, a) = C_0 \int_0^\pi y^a \cos ny \, dy.$$

In the case $-1 < a < 0$, then

$$\begin{aligned} H^*(n, 0, a) &= C_0 \left\{ \int_0^{\pi/n} + \int_{\pi/n}^\pi \right\} y^a \cos ny \, dy \\ &= O(1) \int_0^{\pi/n} y^a \, dy + O(n^{-a-1}) \\ &= O(n^{-a-1}). \end{aligned}$$

Similarly for $1 > a > 0$,

$$\begin{aligned} H^*(n, 0, a) &= C_0 \left\{ [y^a \sin ny/n]_0^\pi - (a/n) \left(\int_0^{\pi/n} + \int_{\pi/n}^\pi \right) y^{a-1} \sin ny \, dy \right\} \\ &= O(n^{-a-1}). \end{aligned}$$

Next, as for (7) we note that

$$\begin{aligned} H^*(n, t, a) &= H^*(n, 0, a) - \int_0^t v^{\alpha+a} (d/dv) g(n, v) \, dv \\ &= \text{etc.} \\ &= H^*(n, 0, a) + O(t^{\alpha+a} n^{\alpha-1}). \end{aligned}$$

This completes the proof of the lemma.

LEMMA 3.8. *Let m be an integer such that $m < w \leq (m+1)$ and suppose $Q_\alpha(t)/t^{\alpha+b+1} \in L(0, 1)$. Then the integral*

$$\int_1^\infty m^{\alpha+b} q_\alpha(m/w) w^{-2} \, dw$$

converges.

Proof. We have: (i) $Q_\alpha(t)$ is positive and increasing over $(0, 1)$ and (ii) $Q_\alpha(t)/t^{\alpha+b+1} \in L(0, 1)$. Then

$$\begin{aligned}
 \int_1^\infty m^{\alpha+b} q_\alpha(m/w) w^{-2} dw &= \sum_{k=1}^\infty \int_k^{k+1} m^{\alpha+b} q_\alpha(m/w) \frac{dw}{w^2} \\
 &\leq K \sum_{k=1}^\infty k^{\alpha+b-1} \int_k^{k+1} m q_\alpha(m/w) w^{-2} dw \\
 &\leq K \sum_{k=1}^\infty k^{\alpha+b-1} \int_{(1-1/k)}^1 q_\alpha(y) dy \\
 &\leq K \sum_{k=1}^\infty k^{\alpha+b-1} Q_\alpha(1/k) \\
 &\leq K \int_1^\infty Q_\alpha(1/u) u^{\alpha+b-1} du \\
 &\leq K \int_0^1 Q_\alpha(z) z^{-(\alpha+b+1)} dz \\
 &< \infty.
 \end{aligned}$$

4. PROOF OF THEOREM 2.1

For $n \geq 1$

$$\begin{aligned}
 \frac{\pi}{2} A_n(x) &= \int_0^\pi \phi(t) \cos nt dt \\
 &= \sum_{r=1}^h n^{r-1} \cos\left(nt - \frac{1}{2}(r-1)\pi\right) \Phi_r(t) \Big|_0^\pi \\
 &\quad + n^h \int_0^\pi \cos\left(nt - \frac{1}{2}h\pi\right) \Phi_h(t) dt,
 \end{aligned}$$

and for $h \geq 0$,

$$\begin{aligned}
 &\Gamma(h+1-\alpha) \int_0^\pi \cos\left(nt - \frac{1}{2}h\pi\right) \Phi_h(t) dt \\
 &= \int_0^\pi \cos\left(nt - \frac{1}{2}h\pi\right) \int_0^t (t-u)^{h-\alpha} d\Phi_\alpha(u) dt \\
 &= \int_0^\pi \int_u^\pi (t-u)^{h-\alpha} \cos\left(nt - \frac{1}{2}h\pi\right) dt d\Phi_\alpha(u)
 \end{aligned}$$

$$\begin{aligned}
&= \left[\Phi_\alpha(u) \int_u^\pi (t-u)^{h-\alpha} \cos\left(nt - \frac{1}{2}h\pi\right) dt \right]_{u=0}^\pi \\
&\quad - \frac{1}{\Gamma(\alpha+1)} \int_0^\pi u^\alpha \phi_\alpha(u) \frac{d}{du} g(n, u) du \\
&= \frac{1}{\Gamma(\alpha+1)} \left[\phi_\alpha(u) \int_u^\pi v^\alpha \frac{d}{dv} g(n, v) dv \right]_0^\pi \\
&\quad - \frac{1}{\Gamma(\alpha+1)} \int_0^\pi \int_u^\pi v^\alpha \frac{d}{dv} g(n, v) dv d\phi_\alpha(u) \\
&= -\phi_\alpha(0)H(n, 0) - \int_0^\pi H(n, u) d\phi_\alpha(u).
\end{aligned}$$

Thus for $\alpha > 0$

$$\begin{aligned}
\frac{\pi}{2} A_n(x) &= \sum_{r=1}^h n^{r-1} \cos\left(n - \frac{1}{2}(r-1)\right) \pi \Phi_r(\pi) \\
&\quad - \frac{1}{\Gamma(h+1-\alpha)} n^h \phi_\alpha(0) H(n, 0) \\
&\quad - \frac{1}{\Gamma(h+1-\alpha)} n^h \int_0^\pi H(n, t) d\phi_\alpha(t).
\end{aligned}$$

By definition, $\Sigma A_n(x) \in |N(q_\alpha)|$ if, and only if

$$I = \int_1^\infty \frac{dw}{w^2} \left| \sum_{n < w} n A_n(x) q_\alpha\left(\frac{n}{w}\right) \right| < \infty.$$

As $\phi_\alpha(t) \in BV(0, \pi)$, in view of Lemma 3.5(i) to prove the theorem it is then sufficient to show that

$$I_{1r} = \int_1^\infty \left| \sum_{n < w} q_\alpha\left(\frac{n}{w}\right) n^r \cos n\pi \right| \frac{dw}{w^2} < \infty, \quad 1 \leq r \leq h \quad (12)$$

and

$$\begin{aligned}
I_2(t) &= \int_1^\infty \left| \sum_{n < w} q_\alpha\left(\frac{n}{w}\right) n^{h+1} H(n, t) \right| \frac{dw}{w^2} \\
&= O(1), \quad \text{for } 0 < t \leq \pi.
\end{aligned} \quad (13)$$

For $1 \leq r \leq h-1$, we refer to Lemma 3.4(i) to get

$$\begin{aligned}
I_{1r} &= O(1) \int_1^\infty \frac{dw}{w^2} \\
&= O(1).
\end{aligned} \quad (14)$$

Also in view of Lemma 3.4(ii) and Lemma 3.2 we obtain

$$\begin{aligned}
 I_{1h} &= \int_1^\infty \left| \sum_{n < w} q_\alpha \left(\frac{n}{w} \right) n^h \cos n\pi \right| \frac{dw}{w^2} \\
 &= O \left(\int_1^\infty \left| q_\alpha^{(h)} \left(1 - \frac{1}{2} \right) \right| \frac{dw}{w^2} \right) + O \left(\int_1^\infty \frac{Q_\alpha(1/w)}{w} dw \right) \\
 &= O \left(\int_0^1 |q_\alpha^{(h)}(y)| dy \right) + O \left(\int_0^1 \frac{Q_\alpha(z)}{z} dz \right) \\
 &= O(1)
 \end{aligned} \tag{15}$$

which completes the proof for (12).

We may note that the case $0 \leq \alpha < 1$ of this theorem is covered by Theorem 2.3 as well as by Theorem 2.4. Hence a proof of this case is obtained from the proof of Theorem 2.3 given ahead. We shall then study here the case $\alpha \geq 1$ only.

For $\alpha \geq 1$, let

$$I_2(t) = \int_1^{\pi/t} + \int_{\pi/t}^\infty \tag{16}$$

$$= I_{21}(t) + I_{22}(t), \quad \text{say.} \tag{17}$$

Using Lemma 3.5(ii) and the results at (14) and (15) we obtain

$$\begin{aligned}
 I_{21}(t) &= K_1 \int_1^{\pi/t} \left| \sum_{n < w} q_\alpha \left(\frac{n}{w} \right) n^{h+1} H(n, 0) \right| \frac{dw}{w^2} \\
 &\quad + K_2 t^\alpha \int_1^{\pi/t} \left| \sum_{n < w} n^\alpha q_\alpha \left(\frac{n}{w} \right) \right| \frac{dw}{w^2} \\
 &= O(1) + K t^\alpha \int_1^{\pi/t} \frac{1}{w^2} \int_1^w u^\alpha q_\alpha \left(\frac{u}{w} \right) du dw \\
 &= O(1) + K t^\alpha \int_1^{\pi/t} u^\alpha \int_u^{\pi/t} q_\alpha \left(\frac{u}{w} \right) \frac{dw}{w^2} \\
 &= O(1) + K t^\alpha \int_1^{\pi/t} u^{\alpha-1} \int_{ut/\pi}^1 q_\alpha(y) dy du \\
 &= O(1), \quad \text{for } 0 < t \leq \pi.
 \end{aligned} \tag{18}$$

For $I_{22}(t)$ we use Lemma 3.5(iii) to get

$$\begin{aligned}
 I_{22} &= \int_{\pi/t}^{\infty} \left| \sum_{n < w} q_{\alpha} \left(\frac{n}{w} \right) n^{h+1} H(n, t) \right| \frac{dw}{w^2} \\
 &\leq K \int_1^{\infty} \left| \sum_{n < w} q_{\alpha} \left(\frac{n}{w} \right) n^{h+1} H(n, 0) \right| \frac{dw}{w^2} \\
 &\quad + \sum_{r=0}^h K_r t^{h-r} \int_{t/\pi}^1 x^{\alpha-h+r-1} (1-x)^{h-\alpha} \\
 &\quad \int_{\pi/t}^{\infty} \left| \sum_{n < w} q_{\alpha} \left(\frac{n}{w} \right) n^{h-r} \cos \left(\frac{nt}{x} - \frac{1}{2} (h-r-1) \pi \right) \right| \frac{dw}{w^2} dx \\
 &\quad + K_{h+1} t^{\alpha} \int_{\pi/t}^{\infty} \left| \int_t^{\pi} (y-t)^{h-\alpha} \sum_{n < w} q_{\alpha} \left(\frac{n}{w} \right) n^{h+1} \right. \\
 &\quad \quad \quad \left. \times \cos \left(ny - \frac{1}{2} h \pi \right) dy \right| \frac{dw}{w^2} \\
 &= I_{221} + I_{222} + I_{223}, \quad \text{say.}
 \end{aligned} \tag{19}$$

From (12), (14), and (15) we have that

$$I_{221} = O(1), \quad \text{for } 0 < t \leq \pi. \tag{20}$$

Lemma 3.4(i)–(ii) and Lemma 3.2 give

$$\begin{aligned}
 I_{222} &= \sum_{r=1}^h K_r t^{h-r} \int_{t/\pi}^1 x^{\alpha-h+r-1} (1-x)^{h-\alpha} x^{h+1-r} t^{-h-1+r} \int_{\pi/t}^{\infty} \frac{dw}{w^2} \\
 &\quad + K_0 t^h \int_{t/\pi}^1 x^{\alpha-h-1} (1-x)^{h-\alpha} \\
 &\quad \times \left\{ \int_{\pi/t}^{\infty} x^{h+1} t^{-h-1} \left| q_{\alpha}^{(h)} \left(1 - \frac{\pi}{tw} \right) \right| \frac{dw}{w^2} \right. \\
 &\quad \quad \quad \left. + K t^h \int_{\pi/t}^{\infty} x^h t^{-h} Q_{\alpha} \left(\frac{\pi}{tw} \right) \frac{dw}{w} \right\} dx \\
 &= \sum_{r=1}^h K_r \int_{t/\pi}^1 x^{\alpha} (1-x)^{h-\alpha} dx t^{-1} \int_{\pi/t}^{\infty} \frac{dw}{w^2} \\
 &\quad + K_0 \int_{t/\pi}^1 x^{\alpha-1} (1-x)^{h-\alpha} dx \\
 &\quad \times \left\{ x t^{-1} \int_0^1 |q_{\alpha}^{(h)}(y)| t dy + K \int_0^1 \frac{Q_{\alpha}(z)}{z} dz \right\} \\
 &= O(1), \quad \text{for } 0 < t \leq \pi.
 \end{aligned} \tag{21}$$

For I_{223} let

$$\begin{aligned} I_{223} &= Kt^\alpha \int_{\pi/t}^\infty \left| \int_t^\pi (y-t)^{h-\alpha} \sum_{n < w} q_\alpha\left(\frac{n}{w}\right) n^{h+1} \cos\left(ny - \frac{1}{2}h\pi\right) dy \right| \frac{dw}{w^2} \\ &= Kt^\alpha \int_{\pi/t}^\infty V(t, w) \left| \frac{dw}{w^2} \right|, \quad \text{say.} \end{aligned} \quad (22)$$

If $(t + \frac{\pi}{w}) < \pi$ we split the integral for V as follows below. If, however, $\pi \leq (t + \frac{\pi}{w})$, we do not need to split the integral and obtain directly the estimates for $V(t, w)$ as worked out over here. Referring to Lemma 3.4(ii)–(iii) we get

$$\begin{aligned} V(t, w) &= \int_t^{t+\pi/w} (y-t)^{h-\alpha} \sum_{n < w} q_\alpha\left(\frac{n}{w}\right) n^{h+1} \cos\left(ny - \frac{1}{2}h\pi\right) dy \\ &\quad + \left[(y-t)^{h-\alpha} \sum_{n < w} q_\alpha\left(\frac{n}{w}\right) n^h \cos\left(ny - \frac{1}{2}(h+1)\pi\right) \right]_{y=t+\pi/w}^\pi \\ &\quad - (h-\alpha) \int_{t+\pi/w}^\pi (y-t)^{h-\alpha-1} \\ &\quad \times \sum_{n < w} q_\alpha\left(\frac{n}{w}\right) n^h \cos\left(ny - \frac{1}{2}(h+1)\pi\right) dy \\ &= \int_t^{t+\pi/w} (y-t)^{h-\alpha} \\ &\quad \times \left\{ K_1 w y^{-h-1} \left| q_\alpha^{(h)}\left(1 - \frac{\pi}{tw}\right) \right| + K_2 w^2 y^{-h} Q_\alpha\left(\frac{\pi}{tw}\right) \right\} dy \\ &\quad + O\left(w^{\alpha-h} \left| q_\alpha^{(h)}\left(1 - \frac{\pi}{tw}\right) \right| t^{-h-1}\right) + O\left(w^{\alpha-h+1} t^{-h} Q_\alpha\left(\frac{\pi}{tw}\right)\right) \\ &= O\left(w^{\alpha-h} t^{-h-1} \left| q_\alpha^{(h)}\left(1 - \frac{\pi}{tw}\right) \right|\right) + O\left(w^{\alpha-h+1} t^{-h} Q_\alpha\left(\frac{\pi}{tw}\right)\right). \end{aligned}$$

In view of Lemma 3.2 we then get

$$\begin{aligned} I_{223} &= O\left(t^{\alpha-h-1} \int_{\pi/t}^\infty w^{\alpha-h} \left| q_\alpha^{(h)}\left(1 - \frac{\pi}{tw}\right) \right| \frac{dw}{w^2} \right) \\ &\quad + O\left(t^{\alpha-h} \int_{\pi/t}^\infty w^{\alpha-h+1} Q_\alpha\left(\frac{\pi}{tw}\right) \frac{dw}{w^2} \right) \end{aligned}$$

$$\begin{aligned}
&= O(1) \int_0^1 \frac{|q_\alpha^{(h)}(u)|}{(1-u)^{\alpha-h}} du + O(1) \int_0^1 \frac{Q_\alpha(z)}{z^{\alpha-h+1}} dz \\
&= O(1), \quad \text{for } 0 < t \leq \pi.
\end{aligned}$$

This terminates the proof of the theorem.

5. PROOF OF THEOREM 2.3

We have for $1 > \alpha \geq 0$

$$\begin{aligned}
\frac{\pi}{2} A_n(x) &= \int_0^\pi \phi(t) \cos nt \, dt \\
&= \frac{1}{\Gamma(1-\alpha)} \int_0^\pi \cos nt \int_0^t (t-u)^{-\alpha} d\Phi_\alpha(u) \, dt \\
&= \frac{1}{\Gamma(1-\alpha)} \int_0^\pi \int_u^\pi (t-u)^{-\alpha} \cos nt \, dt d\Phi_\alpha(u) \\
&= \frac{1}{\Gamma(1-\alpha)} \left[\Phi_\alpha(u) \int_u^\pi (t-u)^{-\alpha} \cos nt \, dt \right]_0^\pi \\
&\quad - \frac{1}{\Gamma(1-\alpha)\Gamma(\alpha+1)} \int_0^\pi u^\alpha \phi_\alpha(u) \frac{d}{du} g(n, u) \, du \\
&= \frac{1}{\Gamma(1-\alpha)\Gamma(\alpha+1)} \left[u^{-\alpha} \phi_\alpha(u) \int_u^\pi v^{\alpha+a} \frac{d}{dv} g(n, v) \, dv \right]_0^\pi \\
&\quad - \frac{1}{\Gamma(1-\alpha)\Gamma(\alpha+1)} \\
&\quad \times \int_0^\pi \int_u^\pi v^{\alpha+a} \frac{d}{dv} g(n, v) \, dv d(u^{-\alpha} \phi_\alpha(u)) \\
&= \frac{1}{\Gamma(1-\alpha)\Gamma(\alpha+1)} C_0 H^*(n, 0, a) \\
&\quad - \frac{1}{\Gamma(1-\alpha)\Gamma(\alpha+1)} \int_0^\pi H^*(n, u, a) d(u^{-\alpha} \phi_\alpha(u)).
\end{aligned}$$

By Lemma 3.7 we note that (i) $H^*(n, 0, 0) = 0$ and (ii) when $a \neq 0$, $\Sigma n^b H^*(n, 0, a)$ converges absolutely. In view of the fact that the method $|N(q_\alpha)|$ is absolutely regular, and that $u^{-\alpha} \phi_\alpha(u) \in BV(0, \pi)$, to show that

$\Sigma n^b A_n \in |N(q_\alpha)|$ it is then sufficient to show that

$$\int_1^\infty \left| \sum_{n < w} q_\alpha \left(\frac{n}{w} \right) n^{b+1} H^*(n, t, a) \right| \frac{dw}{w^2} = O(1), \quad \text{for } 0 < t \leq \pi. \quad (23)$$

However, we note that in the case $(\alpha + b) < 0$, by Lemma 3.7(iii), for $0 < t \leq \pi$, $n^b H^*(n, t, a) = n^b H^*(n, 0, a) + O(n^{\alpha+b-1})$. Therefore

$$\sum n^b H^*(n, t, a)$$

converges absolutely for $0 < t \leq \pi$. Thus we need to consider the case $(\alpha + b) \geq 0$ only.

Now let m be an integer such that $m < w \leq (m + 1)$ and let us write

$$\begin{aligned} & \sum_{n < w} q_\alpha \left(\frac{n}{w} \right) n^{b+1} H^*(n, t, a) \\ &= \sum_{n < w-1} q_\alpha \left(\frac{n}{w} \right) n^{b+1} H^*(n, t, a) + q_\alpha \left(\frac{m}{w} \right) m^{b+1} H^*(m, t, a). \end{aligned} \quad (24)$$

From Lemma 3.7(iii) we have

$$H^*(m, t, a) = H^*(m, 0, a) + O(t^{\alpha+a} m^{\alpha-1}).$$

Therefore, in view of (i) the absolute convergence of $\Sigma n^b H^*(n, 0, a)$ and (ii) Lemma 3.8, to prove the theorem it is enough now to show that

$$\begin{aligned} J(t) &= \int_1^\infty \left| \sum_{n < w-1} q_\alpha \left(\frac{n}{w} \right) n^{b+1} H^*(n, t, a) \right| \frac{dw}{w^2} \\ &= O(1), \quad 0 < t \leq \pi. \end{aligned} \quad (25)$$

Let

$$\begin{aligned} J(t) &= \left(\int_1^{2\pi/t} + \int_{2\pi/t}^\infty \right) \\ &= J_1(t) + J_2(t), \quad \text{say.} \end{aligned} \quad (26)$$

In the case $\alpha = 0$, $a = 0 \geq b \geq -1$, using Lemma 3.7(i) we have

$$\begin{aligned}
 J_1(t) &\leq Kt \int_1^{2\pi/t} \left(\sum_{n < w-1} q_\alpha \left(\frac{n}{w} \right) n^{b+1} \right) \frac{dw}{w^2} \\
 &\leq Kt \int_1^{2\pi/t} \int_1^w u^{b+1} q_\alpha \left(\frac{u}{w} \right) du \frac{dw}{w^2} \\
 &= Kt \int_1^{2\pi/t} w^b \int_{1/w}^1 x^{b+1} q_\alpha(x) dx dw \\
 &= O(1), \quad \text{for } 0 < t \leq \pi.
 \end{aligned} \tag{27}$$

For the other values of α and a using the order estimate in Lemma 3.7(iii), as $a > b$

$$\begin{aligned}
 J_1(t) &= O(1) + O(t^{\alpha+a}) \int_1^{2\pi/t} \left(\sum_{n < w-1} q_\alpha \left(\frac{n}{w} \right) n^{\alpha+b} \right) \frac{dw}{w^2} \\
 &= O(1) + O(t^{\alpha+a}) \int_1^{2\pi/t} w^{\alpha+b-1} \int_{1/w}^1 x^{\alpha+b} q_\alpha(x) dx dw \\
 &= O(1), \quad \text{for } 0 < t \leq \pi.
 \end{aligned} \tag{28}$$

For J_2 we consider first the case $\alpha = 0$, and so $1 > a \geq 0$. Here we have

$$\begin{aligned}
 H^*(n, t, a) &= - \int_t^\pi y^a \cos ny dy \\
 &= t^a \frac{\sin nt}{n} - \frac{a}{n} \int_t^\pi y^{a-1} \sin ny dy,
 \end{aligned}$$

the second term being zero if $a = 0$. Then

$$\begin{aligned}
 J_2(t) &\leq t^a \int_{2\pi/t}^\infty \left| \sum_{n < w-1} q_\alpha \left(\frac{n}{w} \right) n^b \sin nt \right| \frac{dw}{w^2} \\
 &\quad + a \int_{2\pi/t}^\infty \left| \int_t^\pi y^{a-1} \sum_{n < w-1} q_\alpha \left(\frac{n}{w} \right) n^b \sin ny dy \right| \frac{dw}{w^2}.
 \end{aligned} \tag{29}$$

By Lemma 3.6, for $1 > a \geq b \geq -1$, $a > -1$ we get

$$\begin{aligned}
 & t^a \int_{2\pi/t}^{\infty} \left| \sum_{n < w-1} q_{\alpha} \left(\frac{n}{w} \right) n^b \sin nt \right| \frac{dw}{w^2} \\
 &= O(1) + O(t^{a-1}) \int_{2\pi/t}^{\infty} (K_1 w^b + K_2 t^{-b}) q_{\alpha} \left(1 - \frac{\pi}{tw} \right) \frac{dw}{w^2} \\
 &\quad + O(t^a) \int_{2\pi/t}^{\infty} w^{b+1} \left(K_3 + K_4 \left(1 - \frac{\pi}{tw} \right)^b \right) Q_{\alpha} \left(\frac{\pi}{tw} \right) \frac{dw}{w^2} \\
 &= O(1) + O(t^{a-b}) \int_0^1 \frac{q_{\alpha}(w)}{(1-u)^b} du + O(t^{a-b}) \int_0^1 \frac{Q_{\alpha}(z)}{z^{b+1}} dz \\
 &= O(1), \quad \text{for } 0 < t \leq \pi.
 \end{aligned} \tag{30}$$

Similarly for $0 \leq b < a < 1$ we obtain that

$$\int_{2\pi/t}^{\infty} \left| \int_t^{\pi} y^{a-1} \sum_{n < w-1} q_{\alpha} \left(\frac{n}{w} \right) n^b \sin ny dy \right| \frac{dw}{w^2} = O(1), \quad \text{for } 0 < t \leq \pi. \tag{31}$$

From (29) to (31) we complete the proof for

$$J_2 = O(1)$$

in the case $\alpha = 0$.

For $0 < \alpha < 1$, using Lemma 3.7(iv) for H^* we get

$$\begin{aligned}
 J_2(t) &= K_1 \int_{2\pi/t}^{\infty} \int_{t/\pi}^1 x^{\alpha+a-1} (1-x)^{-\alpha} \\
 &\quad \times \left| \int_{t/x}^{\pi} y^a \sum_{n < w-1} q_{\alpha} \left(\frac{n}{w} \right) n^{b+1} \cos ny dy \right| dx \frac{dw}{w^2} \\
 &\quad + K_2 t^{\alpha+a} \int_{2\pi/t}^{\infty} \left| \int_t^{\pi} (y-t)^{-\alpha} \sum_{n < w-1} q_{\alpha} \left(\frac{n}{w} \right) n^{b+1} \cos ny dy \right| \frac{dw}{w^2} \\
 &= J_{21}(t) + J_{22}(t), \quad \text{say,}
 \end{aligned} \tag{32}$$

where we may recall that J_{21} needs to be considered only when $(\alpha + a) > 0$.

We write again

$$\int_{t/x}^{\pi} y^a \cos ny \, dy = t^a \frac{x^{-a}}{n} \sin \frac{nt}{x} - \frac{a}{n} \int_{t/x}^{\pi} y^{a-1} \sin ny \, dy, \quad (33)$$

where the second term is 0 when $a = 0$. We then split J_{21} as $J_{211} + J_{212}$ corresponding with the two terms on the right hand side at (33).

For $1 > a \geq b - 1$, $a > -1$ we get as for (30) that

$$\begin{aligned} J_{211} &= K_1 t^a \int_{2\pi/t}^{\infty} \int_{t/\pi}^1 x^{\alpha-1} (1-x)^{-\alpha} \left| \sum_{n < (w-1)} q_{\alpha}(n/w) n^b \sin \frac{nt}{x} \right| dx \frac{dw}{w^2} \\ &= O(1), \quad \text{for } 0 < t \leq \pi. \end{aligned} \quad (34)$$

Similarly with $a \neq 0$ and $1 > a > b \geq -1$, $a > -1$, we obtain (cf. (31))

$$\begin{aligned} J_{212} &= K_1 \int_{t/\pi}^1 x^{\alpha+a-1} (1-x)^{-\alpha} \int_{t/x}^{\pi} y^{a-1} \\ &\quad \int_{2\pi/t}^{\infty} \left| \sum_{n < (w-1)} q_{\alpha}(n/w) n^b \sin ny \right| \frac{dw}{w^2} dy \, dx \\ &= O(1), \quad \text{for } 0 < t \leq \pi. \end{aligned} \quad (35)$$

Estimates (34) and (35) then give that

$$J_{21} = O(1), \quad \text{for } 0 < t \leq \pi. \quad (36)$$

Finally, referring to the analysis for I_{223} , with apparent modifications in the details given there, in view of Lemma 3.6 again for $1 > a \geq b \geq -1$ we get

$$\begin{aligned} J_{22} &= O(t^{\alpha+a}) \int_{2\pi/t}^{\infty} \left| \int_t^{\pi} (y-t)^{-\alpha} \sum_{n < w-1} q_{\alpha}\left(\frac{n}{w}\right) n^{b+1} \cos ny \, dy \right| \frac{dw}{w^2} \\ &= O(t^{\alpha+a-1}) \int_{2\pi/t}^{\infty} w^{\alpha+b} q_{\alpha}\left(1 - \frac{\pi}{tw}\right) \frac{dw}{w^2} \\ &\quad + O(t^{\alpha+a-b-1}) \int_{2\pi/t}^{\infty} w^{\alpha} q_{\alpha}\left(1 - \frac{\pi}{tw}\right) \frac{dw}{w^2} \\ &\quad + O(t^{\alpha+a}) \int_{2\pi/t}^{\infty} w^{\alpha+b+1} \left(K_1 + K_2 \left(1 - \frac{\pi}{tw}\right)^b\right) Q_{\alpha}\left(\frac{\pi}{tw}\right) \frac{dw}{w^2} \\ &= O(t^{a-b}) \int_0^1 \frac{q_{\alpha}(y)}{(1-y)^{\alpha+b}} dy + O(t^{a-b}) \int_0^1 \frac{Q_{\alpha}(z)}{z^{\alpha+b+1}} dz \\ &= O(1), \quad \text{for } 0 < t \leq \pi. \end{aligned} \quad (37)$$

Collecting now the results from (23)–(37) we complete the proof of Theorem 2.3.

6. PROOF OF THEOREM 2.4

Proof. As for Theorem 2.3 we have that

$$\begin{aligned}
 \frac{\pi}{2} A_n &= \int_0^\pi \phi(t) \cos nt \, dt \\
 &= -\frac{1}{\Gamma(1-\alpha)\Gamma(1+\alpha)} \int_0^\pi u^\alpha \phi_\alpha(u) \frac{d}{du} g(n, u) \, du \\
 &= \frac{1}{\Gamma(1-\alpha)\Gamma(1+\alpha)} \left[\phi_\alpha(u) \int_u^\pi t^\alpha \frac{d}{dt} g(n, t) \, dt \right]_0^\pi \\
 &\quad - \frac{1}{\Gamma(1-\alpha)\Gamma(1+\alpha)} \int_0^\pi \int_u^\pi t^\alpha \frac{d}{dt} g(n, t) \, dt \, d\phi_\alpha(u) \\
 &= \frac{1}{\Gamma(1-\alpha)\Gamma(1+\alpha)} CH(n, 0) \\
 &\quad - \frac{1}{\Gamma(1-\alpha)\Gamma(1+\alpha)} \int_0^\pi H(n, t) \, d\phi_\alpha(t).
 \end{aligned}$$

From Lemma 3.7 we obtain that $H(n, 0) = H^*(n, 0, 0) = 0$. Thus

$$\frac{\pi}{2} A_n(x) = \frac{-1}{\Gamma(1-\alpha)\Gamma(1+\alpha)} \int_0^\pi H(n, t) \, d\phi_\alpha(t).$$

Therefore, $\Sigma A_n(x) n^b \in |N(q_\alpha)|$ if, and only if

$$\int_1^\infty \left| \sum_{n < w} q_\alpha\left(\frac{n}{w}\right) n^{b+1} \int_0^\pi H(n, t) \, d\phi_\alpha(t) \right| \frac{dw}{w^2}$$

converges.

Under the hypothesis $\int_0^\pi t^{-c} |d\phi_\alpha(t)| < \infty$ to prove the theorem then it is sufficient to show that for $0 < t \leq \pi$

$$t^c \int_1^\infty \left| \sum_{n < w} q_\alpha\left(\frac{n}{w}\right) n^{b+1} H(n, t) \right| \frac{dw}{w^2} = O(1).$$

Analysing as in the case of Theorem 2.3 we note that we need to study only the case $\alpha + b \geq 0$. Then again after Lemma 3.8 it is sufficient to show that

$$\begin{aligned} J^0(t) &:= t^c \int_1^\infty \left| \sum_{n < w-1} q_\alpha \left(\frac{n}{w} \right) n^{b+1} H(n, t) \right| \frac{dw}{w^2} \\ &= O(1), \quad \text{for } 0 < t \leq \pi. \end{aligned}$$

In view of the fact that $H(n, t) = H^*(n, t, 0)$ a proof for $J^0(t) = O(1)$ is expected to be less involved than the one for $J(t) = O(1)$. Indeed we find that various analytic steps here are just a repeat and so we shall adopt them from the proof of Theorem 2.3.

Let

$$\begin{aligned} J^0(t) &= \int_1^{2\pi/t} + \int_{2\pi/t}^\infty \\ &= J_1^0(t) + J_2^0(t), \quad \text{say.} \end{aligned}$$

In the case $\alpha = 0$, and so $1 > c \geq 0$, $c \geq b$,

$$\begin{aligned} J_1^0(t) &= O(t^{c+1}) \int_1^{2\pi/t} \left| \sum_{n < w-1} q_\alpha \left(\frac{n}{w} \right) n^{b+1} \right| \frac{dw}{w^2} \\ &= O(1), \quad \text{for } 0 < t \leq \pi, \end{aligned}$$

as for J_1 .

For $1 > \alpha > 0$, using the order estimate from Lemma 3.7(iii),

$$|H(n, t)| = O(t^\alpha n^{\alpha-1})$$

we get, as for J_1 , that

$$\begin{aligned} J_1^0(t) &= O(t^{c+\alpha}) \int_1^{2\pi/t} \left(\sum_{n < w-1} q_\alpha \left(\frac{n}{w} \right) n^{b+\alpha} \right) \frac{dw}{w^2} \\ &= O(1), \quad \text{for } 0 < t \leq \pi. \end{aligned}$$

In the case of J_2^0 it appears that our proof here is much simpler than the one for $J_2(t) = O(1)$, due to the fact that we are dealing with $H^*(n, t, a)]_{a=0}$, that is, with $H(n, t) = H^*(n, t, 0)$.

For the case $\alpha = 0$ and so for $1 > c \geq 0$ we get, as for the corresponding situation of J_2 , that

$$\begin{aligned} J_2^0(t) &= t^c \int_{2\pi/t}^{\infty} \left| \sum_{n < w-1} q_{\alpha} \left(\frac{n}{w} \right) n^b \sin nt \right| \frac{dw}{w^2} \\ &= O(1), \quad 0 < t \leq \pi. \end{aligned}$$

For $0 < \alpha < 1$, we use the expression given in Lemma 3.7(iv) for $H(n, t) \equiv H^*(n, t, 0)$ and thus split the integral for J_2^0 into two parts,

$$\begin{aligned} J_2^0 &= K_1 t^c \int_{2\pi/t}^{\infty} \int_{t/\pi}^1 x^{\alpha-1} (1-x)^{-\alpha} \left| \sum_{n < w-1} q_{\alpha} \left(\frac{n}{w} \right) n^b \sin \frac{nt}{x} \right| \frac{dw}{w^2} \\ &\quad + K_2 t^{c+\alpha} I \int_{2\pi/t}^{\infty} \int_t^{\pi} (y-t)^{-\alpha} \sum_{n < w-1} q_{\alpha} \left(\frac{n}{w} \right) n^{b+1} \cos ny \, dy \left| \frac{dw}{w^2} \right. \\ &= J_{21}^0(t) + J_{22}^0(t), \quad \text{say.} \end{aligned}$$

The expression for J_{22}^0 is practically the same as the one for $J_{22}(t)$. Also the integral for $J_{21}^0(t)$ forms a special situation of the integral for $J_{21}(t)$. Hence we do obtain that

$$J_2^0(t) = O(1), \quad \text{for } 0 < t \leq \pi,$$

and thus complete the proof of the theorem.

REFERENCES

1. L. S. Bosanquet, Notes on the absolute summability (C) of a Fourier series, *J. London Math. Soc.* **11** (1936), 11–15.
2. L. S. Bosanquet, The absolute Cesàro summability of a Fourier series, *Proc. London Math. Soc.* (2) **41** (1936), 517–528.
3. L. S. Bosanquet and H. Kestelman, The absolute convergence of integrals, *Proc. London Math. Soc.* (2) **45** (1939), 88–97.
4. P. Chandra, Absolute summability by Riesz means, *Bull. Calcutta Math. Soc.* **70** (1978), 203–214.
5. G. D. Dikshit, On summability $|R, \exp(w^{\delta}), \gamma|$ of Fourier series, *Analysis* **5** (1985), 1–13.
6. G. D. Dikshit, Absolute Euler summability of Fourier series, *J. Math. Anal. Appl.* **220** (1998), 268–282.
7. G. D. Dikshit and B. Kuttner, On absolute summability of Fourier series by a class of versatile Riesz-means, *Indian J. Math.* **35** (1993), 253–271.
8. K. Matsumoto, On absolute Cesàro summability of a series related to a Fourier series, *Tôhoku Math. J.* (2) **8** (1956), 205–222.
9. R. Mohanty, The absolute Cesàro summability of some series associated with a Fourier series and its allied series, *J. London Math. Soc.* **25** (1950), 63–67.

10. A. F. Moursund, On a method of summation of Fourier series, *Ann. of Math. (2)* **33** (1932), 773–784.
11. A. F. Moursund, On a method of summation of Fourier series, *Ann. of Math. (2)* **34** (1933), 778–798.
12. F. Nevanlinna, Über die Summation der Fourier'schen Reihen und Integrale, *Översikt Finska Vetenskaps-Societetens Förhandlingar A* **64**, No. 3 (1921–1922).
13. B. K. Ray and A. K. Sahoo, On the absolute Nevanlinna summability factors of Fourier series and conjugate series, *J. Indian Math. Soc.* **61** (1995), 161–88.
14. B. K. Ray and M. Samal, Application of the absolute N_q -method to some series and integrals, *J. Indian Math. Soc.* **44** (1980), 217–236.
15. M. Samal, On the absolute N_q -summability of some series associated with Fourier series, *J. Indian Math. Soc.* **50** (1986), 191–209.